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A Class of Complex-valued Harmonic Functions Defined by Dziok-Srivastava Operator

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Abstract: The Dziok-Srivastava [6] operator introduced in the study of analytic functions and associated with generalized hypergeometric functions has been extended to harmonic mappings [2, 12]. Using this operator we introduce a subclass of the class $\mathcal{H}$ of complex-valued harmonic univalent functions $f = h + \overline{g}$ where h is the analytic part and g is the co-analytic part of f in $|z| < 1$. Coefficient bounds,extreme points, inclusion results and closure under an integral operator for this class are obtained.

Keywords: Harmonic functions, Hypergeometric functions, Dziok-Srivastava operator, extreme points, integral operator

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1 Introduction

Harmonic mappings have found applications in many diverse fields such as engineering, aerodynamics and other branches of applied mathematics. Harmonic mappings in a domain $D \subseteq \mathcal{C}$ are univalent complex-valued harmonic functions $f = u + iv$ where both u and v are real harmonic. The important work of Clunie and Sheil-Small [5] on the class consisting of complex-valued harmonic orientation-preserving univalent functions f defined on the open unit disk $\mathcal{U}$ formed the basis for several investigations on different subclasses of harmonic univalent functions (See for example [1] and references therein).

In any simply-connected domain D it is known that [5] we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [5]).

Denote by $\mathcal{H}$ the family of harmonic functions

$$f = h + \overline{g} \tag{1}$$

which are univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the analytic functions h and g are given by

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \;\; g(z) = \sum_{m=1}^{\infty} b_m z^m.$$

Hence

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}, \;\; |b_1| < 1. \tag{2}$$

We note that the family $\mathcal{H}$ reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is $g \equiv 0$.

For complex numbers $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, q)$ the generalized hypergeometric function [13] ${}_pF_q(z)$ is defined by

$${}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z) := \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \, \frac{z^m}{m!}, \tag{3}$$

$$(p \leq q + 1; \;\; p, q \in \;\; N_0 := \;\; N \cup \{0\}; z \in \mathcal{U})$$

where N denotes the set of all positive integers and $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \begin{cases} 1, & m = 0 \\ a(a+1)(a+2)\dots(a+m-1), & m \in N. \end{cases} \tag{4}$$

Dziok and Srivastava [6] introduced an operator in their study of analytic functions associated with generalized hypergeometric functions. This Dziok-Srivastava operator is known to include many well-known operators as special cases.

Let

$$H(\alpha_1, \dots \alpha_p; \beta_1, \dots, \beta_q) : A \to A$$

be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \dots \alpha_p; \beta_1, \dots, \beta_q))(\phi)](z) &:= z\ {}_pF_q(\alpha_1, \alpha_2, \dots \alpha_p; \beta_1, \beta_2 \dots, \beta_q; z) * \phi(z) \\ &= z + \sum_{m=2}^{\infty} \Gamma_m\ a_m\ z^m, \end{aligned} \tag{5}$$

where

$$\Gamma_m = \frac{(\alpha_1)_{m-1} \dots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \dots (\beta_q)_{m-1}} \frac{1}{(m-1)!} \tag{6}$$

and $\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q$ are positive real numbers,such that $p \leq q+1; p, q \in \mathbb{N} \cup \{0\}$, and $(a)_m$ is the familiar Pochhammer symbol.

The linear operator $H(\alpha_1, \dots \alpha_p; \beta_1, \dots, \beta_q)$ or $H_q^p[\alpha_1, \beta_1]$ in short, is the Dziok-Srivastava operator (see [6] and [17]), which includes several well known operators.

The Dziok-Srivastava operator when extended to the harmonic function $f = h + \overline{g}$ is defined by

$$H_q^p[\alpha_1, \beta_1]f(z) = H_q^p[\alpha_1, \beta_1]h(z) + \overline{H_q^p[\alpha_1, \beta_1]g(z)} \tag{7}$$

Motivated by earlier works of [4, 7, 8, 9, 10, 11, 14, 16, 18] on harmonic functions, we introduce here a new subclass $G_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ of $\mathcal{H}$ using the Dziok-Srivastava operator extended to harmonic functions.

Let $G_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ denote the subfamily of starlike harmonic functions $f \in \mathcal{H}$ of the form (1) such that

$$\operatorname{Re}\left\{ 1 + (1 + e^{i\psi}) \frac{\left[\begin{array}{l} z^2(H_q^p[\alpha_1, \beta_1]h(z))'' \\ + \overline{2z(H_q^p[\alpha_1, \beta_1]g(z))' + z^2(H_q^p[\alpha_1, \beta_1]g(z))''} \end{array} \right]}{z(H_q^p[\alpha_1, \beta_1]h(z))' - \overline{z(H_q^p[\alpha_1, \beta_1]g(z))'}} \right\} \geq \gamma \tag{8}$$

where $H_q^p[\alpha_1,\beta_1]f(z)$ is defined by (7) $0 \le \gamma < 1$, $z \in \mathcal{U}$ and ψ real.

We also let $T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma) = G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma) \bigcap T_{\mathcal{H}}$ where $T_{\mathcal{H}}$ [16], is the class of harmonic functions f such that

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m - \overline{\sum_{m=1}^{\infty} |b_m| z^m}, \quad |b_1| < 1. \tag{9}$$

We obtain a sufficient coefficient condition for functions f given by (2) to be in the class $G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ and show that this coefficient condition also is necessary for functions belonging to the class $T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$. Also, extreme points for functions in $T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ and certain inclusion results are obtained.

2 Coefficient Condition for the Class $G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$

A sufficient coefficient condition for functions belonging to the class $G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ is now derived.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{m=1}^{\infty} m \left(\frac{2m-1-\gamma}{1-\gamma} |a_m| + \frac{2m+1+\gamma}{1-\gamma} |b_m| \right) \Gamma_m \le 2. \tag{10}$$

$0 \le \gamma < 1$, *then* $f \in G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$.

Proof. When the condition (10) holds for the coefficients of $f = h + \overline{g}$, it is shown that the inequality (8) is satisfied. Write the left side of inequality (8) as

$$\mathrm{Re} \left\{ \frac{\begin{array}{c} z(H_q^p[\alpha_1,\beta_1]h(z))' + (1+e^{i\psi}) z^2 (H_q^p[\alpha_1,\beta_1]h(z))'' \\ + (1+2e^{i\psi}) \overline{z(H_q^p[\alpha_1,\beta_1]g(z))'} + (1+e^{i\psi}) \overline{z^2 (H_q^p[\alpha_1,\beta_1]g(z))''} \end{array}}{z(H_q^p[\alpha_1,\beta_1]h(z))' - \overline{z(H_q^p[\alpha_1,\beta_1]g(z))'}} \right\}$$

$$= \mathrm{Re}\, \frac{A(z)}{B(z)}.$$

Since Re $(w) \ge \gamma$ if and only if $|1-\gamma+w| \ge |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \ge 0. \tag{11}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (11), we get

$$\begin{aligned}
& |A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \\
\geq\ & (2-\gamma)|z|-\sum_{m=2}^{\infty} m(2m-\gamma)\Gamma_m|a_m|\ |z|^m-\sum_{m=1}^{\infty} m(2m+\gamma)\Gamma_m|b_m|\ |\ |z|^m \\
& -\gamma|z|-\sum_{m=2}^{\infty} m(2m-2-\gamma)\Gamma_m|a_m|\ |z|^m-\sum_{m=1}^{\infty} m(2m+2+\gamma)\Gamma_m|b_m|\ |z|^m. \\
\geq\ & 2(1-\gamma)|z|\left\{1-\sum_{m=2}^{\infty} m\frac{2m-1-\gamma}{1-\gamma}\Gamma_m|a_m|-\sum_{m=1}^{\infty} m\frac{2m+1+\gamma}{1-\gamma}\Gamma_m|b_m|\right\} \\
\geq\ & 0
\end{aligned}$$

by inequality (10), which implies that $f\in G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$. □

Now we obtain the necessary and sufficient condition for the function $f=h+\overline{g}$ given by (9) to be in $T_{\mathcal{H}}$.

Theorem 2.2. *Let $f=h+\overline{g}$ be given by (9). Then $f\in T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ if and only if*

$$\sum_{m=1}^{\infty} m\left\{\frac{2m-1-\gamma}{1-\gamma}|a_m|+\frac{2m+1+\gamma}{1-\gamma}|b_m|\right\}\Gamma_m\leq 2 \tag{12}$$

where $0\leq\gamma<1$.

Proof. Since $T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)\subset G_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$, we only need to prove the necessary part of the theorem. Assume that $f\in T_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$, then by virtue of (7) to (8), we obtain

$$\operatorname{Re}\left\{(1-\gamma)+(1+e^{i\psi})\frac{\left[\begin{array}{l} z^2(H_q^p[\alpha_1,\beta_1]h(z))'' \\ +\overline{2z(H_q^p[\alpha_1,\beta_1]g(z))'+z^2(H_q^p[\alpha_1,\beta_1]g(z))''}\end{array}\right]}{z(H_q^p[\alpha_1,\beta_1]h(z))'-\overline{z(H_q^p[\alpha_1,\beta_1]g(z))'}}\right\}\geq 0.$$

The above inequality is equivalent to

$$
\begin{aligned}
&\mathrm{Re}\left\{\frac{z-\left(\begin{array}{l}\sum_{m=2}^{\infty} m[m(1+e^{i\psi})-\gamma-e^{i\psi}]\Gamma_m|a_m|z^m \\ +\sum_{m=1}^{\infty} m[m(1+e^{i\psi})+\gamma+e^{i\psi}]\Gamma_m|b_m|\overline{z}^m\end{array}\right)}{z-\sum_{m=2}^{\infty} m\Gamma_m|a_m|z^m+\sum_{m=2}^{\infty} m\Gamma_m|b_m|\overline{z}^m}\right\} \\
=\ &\mathrm{Re}\left\{\frac{\begin{array}{l}(1-\gamma)-\sum_{m=2}^{\infty} m[m(1+e^{i\psi})-e^{i\psi}-\gamma]\Gamma_m|a_m|z^{m-1} \\ -\frac{\overline{z}}{z}\sum_{m=1}^{\infty} m[m(1+e^{i\psi})+e^{i\psi}+\gamma]\Gamma_m|b_m|\overline{z}^{m-1}\end{array}}{1-\sum_{m=2}^{\infty} m\Gamma_m|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum_{m=1}^{\infty} m\Gamma_m|b_m|\overline{z}^{m-1}}\right\} \\
\geq\ & 0.
\end{aligned}
$$

This condition must hold for all values of $z \in \mathcal{U}$ and for real ψ, so that on taking $z = r < 1$ and $\psi = 0$, the above inequality reduces to

$$
\frac{(1-\gamma)-\left[\sum_{m=2}^{\infty} m(2m-1-\gamma)\Gamma_m|a_m|r^{m-1}+\sum_{m=1}^{\infty} m(2m+1+\gamma)\Gamma_m|b_m|r^{m-1}\right]}{1-\sum_{m=2}^{\infty}\Gamma_m|a_m|r^{m-1}+\sum_{m=1}^{\infty}\Gamma_m|b_m|r^{m-1}} \geq 0. \tag{13}
$$

Letting $r \to 1^-$ through real values, we obtain the condition (12). This completes the proof of Theorem 2.2. □

3 Extreme Points and Inclusion Results

We determine the extreme points of closed convex hulls of $\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ denoted by $clco\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$.

Theorem 3.1. *A function* $f(z) \in clco\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ *if and only if* $f(z) =$

$\sum\limits_{m=1}^{\infty}(X_m h_m(z) + Y_m g_m(z))$ *where*

$$h_1(z) = z, h_m(z) = z - \frac{1-\gamma}{m(2m-1-\gamma)\Gamma_m} z^m;\ \ (m \geq 2),$$
$$g_m(z) = z - \frac{1-\gamma}{m(2m+1+\gamma)\Gamma_m} \overline{z}^m;\ (m \geq 2),$$
$$\sum_{m=1}^{\infty}(X_m + Y_m) = 1, \quad X_m \geq 0 \ \textit{and} \ \ Y_m \geq 0.$$

In particular, the extreme points of $\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ *are* $\{h_m\}$ *and* $\{g_m\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_m(z)) \\ &= \sum_{m=1}^{\infty}(X_m + Y_m)z - \sum_{m=2}^{\infty}\frac{1-\gamma}{m(2m-1-\gamma)\Gamma_m} X_m z^m \\ &\quad - \sum_{m=1}^{\infty}\frac{1-\gamma}{m(2m+1+\gamma)\Gamma_m} Y_m \overline{z}^m \\ &= z - \sum_{m=2}^{\infty} A_m z^m - \sum_{m=1}^{\infty} B_m \overline{z}^m \end{aligned}$$

where $\quad A_m = \dfrac{1-\gamma}{m(2m-1-\gamma))\Gamma_m} X_m,$ and $B_m = \dfrac{1-\gamma}{m(2m+1+\gamma)\Gamma_m} Y_m.$

Therefore

$$\begin{aligned} &\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\Gamma_m}{1-\gamma} A_m + \sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\Gamma_m}{1-\gamma} B_m \\ &= \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m \\ &= 1 - X_1 \leq 1, \end{aligned}$$

and hence $f(z) \in clco\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$.
Conversely, suppose that $f(z) \in clco\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$. Setting

$$X_m = \frac{m(2m-1-\gamma)\Gamma_m}{1-\gamma} A_m, \quad (m \geq 2),\ Y_m = \frac{m(2m+1+\gamma)\Gamma_m}{1-\gamma} B_m,\ (m \geq 1)$$

where $\sum\limits_{m=1}^{\infty}(X_m+Y_m)=1$. Then

$$\begin{aligned}
f(z) &= z-\sum_{m=2}^{\infty}A_m z^m-\sum_{m=1}^{\infty}B_m\overline{z}^m, \quad A_m,\ B_m\geq 0.\\
&= z-\sum_{m=2}^{\infty}\frac{1-\gamma}{m(2m-1-\gamma)\Gamma_m}X_m z^m-\sum_{m=1}^{\infty}\frac{1-\gamma}{m(2m+1+\gamma)\Gamma_m}Y_m\overline{z}^m\\
&= z+\sum_{m=2}^{\infty}(h_m(z)-z)X_m+\sum_{m=1}^{\infty}(g_m(z)-z)Y_m\\
&= \sum_{m=1}^{\infty}(X_m h_m(z)+Y_m g_m(z))
\end{aligned}$$

as required. □

Now we show that $\mathcal{T}_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ is closed under convex combinations of its members.

Theorem 3.2. *The family $\mathcal{T}_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ is closed under convex combinations.*

Proof. For $i=1,2,\ldots,$ suppose that $f_i\in\mathcal{T}_{\mathcal{H}}([\alpha_1,\beta_1],\gamma)$ where

$$f_i(z)=z-\sum_{m=2}^{\infty}a_{i,m}z^m-\sum_{m=2}^{\infty}b_{i,m}\overline{z}^m.$$

Then, by inequality (12)

$$\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\Gamma_m}{(1-\gamma)}a_{i,m}+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\Gamma_m}{(1-\gamma)}b_{i,m}\leq 1. \tag{14}$$

For $\sum\limits_{i=1}^{\infty}t_i=1,\ 0\leq t_i\leq 1,$ the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_i f_i(z)=z-\sum_{m=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i a_{i,m}\right)z^m-\sum_{m=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i b_{i,m}\right)\overline{z}^n.$$

Using the inequality (12), we obtain

$$\begin{aligned}
&\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\Gamma_m}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_i a_{i,m}\right)+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\Gamma_m}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_i b_{i,m}\right)\\
&=\sum_{i=1}^{\infty}t_i\left(\sum_{m=2}^{\infty}\frac{m(2m-1-\gamma)\Gamma_m}{(1-\gamma)}a_{i,m}+\sum_{m=1}^{\infty}\frac{m(2m+1+\gamma)\Gamma_m}{(1-\gamma)}b_{i,n}\right)\\
&\leq\sum_{i=1}^{\infty}t_i=1,
\end{aligned}$$

and therefore $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{T}_{\mathcal{H}}([\alpha_1], \gamma)$. $\square$

Theorem 3.3. *For* $0 \leq \delta \leq \gamma < 1$, *let* $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ *and* $F(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \delta)$. *Then* $f(z) * F(z) \in \mathcal{G}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma) \subset \mathcal{G}_{\mathcal{H}}([\alpha_1, \beta_1], \delta)$.

Proof. Let $f(z) = z - \sum\limits_{m=2}^{\infty} a_m z^m - \sum\limits_{m=1}^{\infty} \bar{b}_m \bar{z}^n \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ and $F(z) = z - \sum\limits_{m=2}^{\infty} A_m z^m - \sum\limits_{m=1}^{\infty} \overline{B}_m \bar{z}^n \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \delta)$. Then $f(z) * F(z) = z + \sum\limits_{m=2}^{\infty} a_m A_m z^m + \sum\limits_{m=1}^{\infty} \bar{b}_m \overline{B}_m \bar{z}^n$.

We note that $|A_m| \leq 1$ and $|B_m| \leq 1$. Now we have

$$\begin{aligned} & \sum_{m=2}^{\infty} \frac{m(2m-1-\delta)\Gamma_m}{1-\delta} |a_m| \quad |A_m| + \sum_{m=1}^{\infty} \frac{m(2m+1+\delta)\Gamma_m}{1-\delta} |b_m| \quad |B_m| \\ \leq & \sum_{m=2}^{\infty} \frac{m(2m-1-\delta)\Gamma_m}{1-\delta} |a_m| + \sum_{m=1}^{\infty} \frac{m(2m+1+\delta))\Gamma_m}{1-\delta} |b_m| \\ \leq & \sum_{m=2}^{\infty} \frac{m(2m-1-\gamma))\Gamma_m}{1-\gamma} |a_m| + \sum_{m=1}^{\infty} \frac{m(2m+1+\gamma))\Gamma_m}{1-\gamma} |b_m| \leq 1, \end{aligned}$$

using Theorem 2.2 since $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ and $0 \leq \delta \leq \gamma < 1$. This proves that $f(z) * F(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \delta)$. $\square$

4 Integral Operator

Now, we examine a closure property of the class $\mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$ under the generalized Bernardi-Libera -Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, c > -1.$$

Theorem 4.1. *Let* $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$. *Then* $L_c(f(z)) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$

Proof. From the representation of $L_c(f(z))$, it follows that

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt.$$

$$= \quad \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{m=2}^{\infty} a_m t^n\right)dt - \overline{\int_0^z t^{c-1}\left(\sum_{m=1}^{\infty} b_m t^n\right)dt}\right)$$

$$= z - \sum_{m=2}^{\infty} A_m z^m - \sum_{n=21}^{\infty} B_m z^m$$

where

$$A_m = \frac{c+1}{c+n}\, a_m; B_m = \frac{c+1}{c+n}\, b_m.$$

Therefore,

$$\begin{aligned}
&\sum_{m=1}^{\infty} m\left(\frac{2m-1-\gamma}{1-\gamma}(\frac{c+1}{c+n}|a_m|) + \frac{2m+1+\gamma}{1-\gamma}(\frac{c+1}{c+n}|b_m|)\right)\Gamma_m \\
\leq\; & \sum_{m=1}^{\infty} m\left(\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\right)\Gamma_m \\
\leq\; & 2(1-\gamma).
\end{aligned}$$

Since $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$, therefore by Theorem 2.2 , $L_c(f(z)) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, \beta_1], \gamma)$. □

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